JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **193**, 431–446 (1995)

Unbounded Translation Invariant Operators on Locally Compact Abelian Groups

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Submitted by C. Foias

Received June 21, 1993

1. INTRODUCTION

We are concerned with L_p theory of closed translation invariant operators (also known as unbounded multipliers) on locally compact Abelian groups. Among other results we characterize such operators on L_1 and L_2 in terms of Fourier multiplication operators. We also consider their relationships to translation invariant operators on other L_p spaces. Though translation invariant operators are characterized in terms of multiplication operators, they exhibit an even better behavior than these. In particular, we are able to prove that any closable translation invariant operator has a uniquely determined closed extension on L_1 and L_2. The corresponding assertion, however, is not true for multiplication operators in general (see Prop. 1).

Having outlined the scope of this paper, let us mention the range of known results: First, in the case of bounded translation invariant operators, virtually all results presented here are well known. Of course, due to the fact that domain problems are not present in the bounded case, some of our results are interesting in the unbounded case only. Second, there are a few papers on unbounded translation invariant operators on L_1 [1, 15, 8]. In particular, the main theorems proved in [1, 15] imply the L_1 characterization result. Also, Theorem 2.3 in [15] shows that on L_1 the closed extension is unique. Furthermore, certain subclasses of translation invariant operators, especially generators of positive C_0-semigroups have been studied thoroughly ([3]; see also [2, 9, 10]).

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0022-247X/95 $12.00

We note that in the present context, the most important difference between L_1 and L_p ($p > 1$) is the fact that L_1 is an algebra w.r.t. convolution, while L_p, in general, is not.

2. Definition and Basic Tools

Throughout we let $(G, +)$ denote a (Hausdorff) locally compact Abelian group with Haar measure λ_G. $\hat{G}$ will be its dual group, i.e., the group of all bounded characters on G. The Haar measures on G and $\hat{G}$, λ_G and $\lambda_{\hat{G}}$, are assumed to be chosen in a way that the inversion theorem holds. (Our general references on these topics are [4, 12].)

We will abbreviate $L_p(G, \lambda_G)$ by $L_p(G)$ (or even L_p if the group is understood) and denote its norm by $\|\cdot\|_p$; $\|\cdot\|_{p,q}$ will be the operator norm on $L(L_p, L_q)$. $C(G)$, $C_b(G)$, $C_c(G)$ denote the spaces of continuous, bounded continuous, and compactly supported continuous $\mathbb{C}$-valued functions on G, respectively. The operator of translation by $g \in G$ is written as τ_g, i.e., $(\tau_g f)(x) = f(x - g)$. $\mathcal{F}$ will stand for the Fourier transform on any of the spaces L_p $(1 \le p \le 2)$. We will usually write $\hat{f}$ for $\mathcal{F}f$. κ denotes reflection, i.e., $(\kappa f)(x) = f(-x)$.

Definition 1. Let $1 \le p < \infty$ and let $T\colon D(T) \subseteq L_p(G) \to L_p(G)$ be a densely defined linear operator. We say that T *commutes with translations* if and only if $\tau_a(D(T)) \subseteq D(T)$ and $T\tau_a f = \tau_a Tf$ for all $f \in D(T)$ and $a \in G$.

T is called *translation invariant* if and only if T is closable and commutes with translations.

The set of translation invariant operators will be denoted by $\mathcal{T}(L_p)$. The subsets of closed and bounded translation invariant operators are denoted by $\mathcal{T}_c(L_p)$ and $\mathcal{T}_b(L_p)$, respectively.

The following basic examples show that we are talking about a real subject:

(i) For arbitrary $p < \infty$, every constant coefficient PDO defined on C_c^∞ is a translation invariant operator w.r.t. the additive group $\mathbb{R}^n$.

(ii) For arbitrary $p < \infty$ the mapping $C^1(\mathbb{T}) \ni f \mapsto f'$ is a translation invariant operator on L_p w.r.t. the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Additional examples of translation invariant operators will become obvious later from the characterizations of translation invariant operators. We note, however, that unbounded translation invariant operators need not exist for general groups and arbitrary p. A trivial example is provided by finite groups. However, even on infinite discrete groups, any translation invariant operator on L_1 is bounded; see Corollary 1 below.

The following lemmas are key tools for the subsequent development.

LEMMA 1. *Let* $1 \le p, q, r \le \infty$ *satisfy* $1/p + 1/q - 1/r = 1$. *If* $A \subseteq L_p$ *and* $B \subseteq L_q$ *are dense subsets, then* $\mathrm{lin}[A * B]$, *the linear hull of* $A * B$, *is weakly*-dense in* L_r, *hence norm dense if* $r < \infty$.

Proof. The fact that $A * B$ is contained in L_r is due to Young's inequality. By the condition imposed on p, q, and r we may assume without restriction that $p < \infty$. Let $f \in L_{r'}$, $g \in B$. Then $f * \kappa g \in L_{p'}$, and

$$0 = \langle f, h * g\rangle = (f * \kappa g * \kappa h)(0) = \langle f * \kappa g, h\rangle$$

for all $h \in A$ implies $f * \kappa g = 0$ a.e.. Hence $h * f * \kappa g = 0$ for all $h \in C_c(G)$. Since $h * f * \kappa g$ is continuous it follows that

$$0 = (h * f * \kappa g)(0) = \langle h * f, g\rangle$$

for all $h \in C_c(G)$ and $g \in B$. But also $h \in C_c(G) \subseteq L_q$, so $h * f \in L_{p'}$, and thus $h * f = 0$. Now, considering $C_c(G)$ as a weakly*-dense subset of L_r, this implies $f = 0$ by the same argument. ∎

The next lemma reveals some basic properties of translation invariant operators.

LEMMA 2. *Let* $1 \le p < \infty$ *and let* $T \in \mathcal{T}(L_p)$. *Then*

(i) $L_1 * D(T) \subseteq D(\overline{T})$, *and* $\overline{T}(g * f) = g * \overline{T}f$ *for all* $g \in L_1$ *and* $f \in D(T)$.

(ii) *If* C *is a core for* T (*i.e.*, $\overline{T}$ *is the closure of* $T|_C$) *and* $D \subseteq L_1$ *is dense then* $\mathrm{lin}[D * C]$ *is a core for* T.

(iii) *If* $1 \le p \le 2$ *and* $D(\overline{T}) \cap L_1$ *is a core for* $\overline{T}$ *then its dual operator* T^* *on* $L_{p'}$ *is an extension of the translation invariant operator*

$$\kappa T\kappa : \kappa(\mathrm{lin}[C_c(G) * (D(\overline{T}) \cap L_1)]) \subseteq L_{p'} \to L_{p'}.$$

Proof. (i) Let $h \in D(T^*)$ and $f \in D(T)$. Then for all $x \in G$ one has

$$|(h * \kappa Tf)(x)| = |(h * \kappa T(\tau_{-x}f))(0)| = |\langle h, T\tau_{-x}f\rangle| \le \|T^*h\|_{p'}\|\tau_{-x}f\|_p,$$

in other words, $\|h * \kappa Tf\|_\infty \le \|T^*h\|_{p'}\|f\|_p$. For $g \in L_1$ Young's inequality yields

$$\langle g * h, Tf\rangle = (g * h * \kappa Tf)(0) \le \|g * h * \kappa Tf\|_\infty \le \|g\|_1\|T^*h\|_{p'}\|f\|_p,$$

hence $g * h \in D(T^*)$, by definition. On the other hand, since $(h * \kappa Tf)(x) = (T^*h * \kappa f)(x)$ we obtain

$$\begin{aligned}\langle T^*(g*h),f\rangle &= \langle g*h, Tf\rangle = (g*h*Tf)(0)\\ &= (g*T^*h*\kappa f)(0) = \langle g*T^*h,f\rangle,\end{aligned}$$

for all $f \in D(T)$, and, by density, $T^*(g*h) = g*T^*h$. This yields

$$\begin{aligned}\langle T^*h, g*f\rangle &= (T^*h*\kappa f*\kappa g)(0)\\ &= (h*(\kappa Tf)*\kappa g)(0)\\ &= h*\kappa(g*Tf)(0)\\ &= \langle h, g*Tf\rangle.\end{aligned}$$

Since $\overline{T}$ is the $\sigma(L_{p'}, L_p)$-dual of T^* (see, e.g., [13, Cor. IV.7.1]) one has $g*f \in D(\overline{T})$ and

$$\langle h, g*Tf\rangle = \langle T^*h, g*f\rangle = \langle h, \overline{T}(g*f)\rangle.$$

By $\sigma(L_{p'}, L_p)$-density of $D(T^*)$ this yields (i).

(ii) The subspace $\operatorname{lin}[D*C]$ is dense by Lemma 1, and by part (i) $\overline{T}|_{\operatorname{lin}[D*C]}$ is a densely defined closable operator on L_p. To show that its closure is an extension of T, hence equals $\overline{T}$, note that for any $f \in D(T)$ there is a sequence $(f_n) \subseteq C$ such that $f_n \to f$ and $Tf_n \to \overline{T}f$ in L_p-norm. Now, there is a sequence $(h_n) \subseteq L_1$ with

$$\lim_{n\to\infty} \|h_n * f_n - f_n\|_p = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|h_n * Tf_n - Tf_n\|_p = 0$$

[4, Thm. 20.15]. Since D is assumed to be dense, and since (f_n) and (Tf_n) are bounded, we may choose (h_n) from D, proving (ii).

(iii) Let $D := \operatorname{lin}[C_c(G) * (D(\overline{T}) \cap L_1)] \subseteq D(\overline{T})$, which is a dense subspace by Lemma 1. Since $p' \geq p$ we also know that D is a weakly*-dense subspace of $L_{p'}$ (norm dense if $p > 1$). Now, if $f \in \kappa D$ and $g \in D(\overline{T}) \cap L_1$, part (i) yields

$$\langle f, \overline{T}g\rangle = (\kappa f * Tg)(0) = \overline{T}(\kappa f * g)(0) = (\overline{T}\kappa f * g)(0) = \langle \kappa\overline{T}\kappa f, g\rangle, \qquad (1)$$

and $\kappa\overline{T}\kappa f \in L_{p'}$, hence $f \in D(T^*)$, since $D(\overline{T}) \cap L_1$ is a core. Finally, (1) also implies $T^*f = \kappa\overline{T}\kappa f$. ∎

3. Characterization on L_1 and L_2

Now we are concerned with characterizations of translation invariant operators in terms of Fourier multiplication operators. The first theorem is the basic result for the L_1-case. Though it is essentially known, we include a sketch of its proof.

THEOREM 1 [1, 15]. *Let $T: D(T) \subseteq L_1 \to L_1$ be a densely defined linear operator. Then the following are equivalent:*

(i) *T is translation invariant.*

(ii) *T is closable, $g * f \in D(\overline{T})$, and $\overline{T}(g * f) = g * Tf$ for all $g \in L_1, f \in D(T)$.*

(iii) *There is a uniquely determined $m_T \in C(\hat{G})$ such that $\widehat{Tf} = m_T \hat{f}$ for all $f \in D(T)$.*

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 2; (iii) $\Rightarrow$ (i) is easy to prove.

For the proof of (ii) $\Rightarrow$ (iii) note that by density $D(T)$ cannot be contained in a maximal ideal and hence for any $\xi \in \hat{G}$ one has $\hat{f}(\xi) \neq 0$ for some $f \in D(T)$. The identity

$$g * Tf = \overline{T}(g * f) = (Tg) * f$$

for $f, g \in D(T)$ yields that

$$m_T(\xi) := \frac{\widehat{Tf}(\xi)}{\hat{f}(\xi)}, \qquad \text{whenever } \hat{f}(\xi) \neq 0,$$

is independent of f. ∎

The function m_T given by (iii) of Theorem 1 will be called the *symbol* of T; we will use the notation

$$\mathcal{F}\mathcal{T}(L_1) := \{m_T \in C(\hat{G}) : m_T \text{ is the symbol of some } T \in \mathcal{T}(L_1)\}.$$

Theorem 1 does not assert that every continuous function is the symbol of a translation invariant operator. In general one only knows

$$\mathcal{F}\mathcal{T}(L_1) \subseteq C(\hat{G}). \tag{2}$$

Indeed, on compact groups equality holds in (2) by part (i) of the following corollary. On the other hand, part (iii) shows that equality does not hold in general.

COROLLARY 1. (i) *If G is compact then any function $\phi: \hat{G} \to \mathbb{C}$ is the symbol of a translation invariant operator $T \in \mathcal{T}(L_1)$.*

(ii) *If G is discrete then $\mathcal{T}(L_1) = \mathcal{T}_b(L_1)$, i.e., every translation invariant operator is bounded, namely, $Tf = g * f$ for some $g \in L_1$.*

(iii) *There is a continuous function on $\mathbb{T}$ which is not the symbol of a translation invariant operator in $\mathcal{T}L_1(\mathbb{Z})$.*

Proof. (i) Recall that the dual group $\hat{G}$ is discrete (cf., e.g., [12], Thm. 1.2.5]), so any function on $\hat{G}$ is continuous. Since $C_c(\hat{G})$ is dense in $L_2(\hat{G})$ and the Fourier transform is isometric, $D := \mathcal{F}^{-1}(C_c(\hat{G}))$ is dense in $L_2(G)$. Since G is a finite measure space, $L_2(G)$ is densely contained in $L_1(G)$, hence D is dense in $L_1(G)$ too. This argument also shows that $\phi\hat{f} \in C_c(\hat{G}) \subseteq \mathcal{F}L_1(G)$, whenever $f \in D$. Hence $Tf := \mathcal{F}^{-1}(\phi\hat{f})$ defines an operator $T \in \mathcal{T}(L_1)$ by Theorem 1.

(ii) Without loss of generality we may assume that T is closed. Once we have shown that Dirac's δ_0 belongs to $D(T)$ we know that $D(T) = L_1$, since $D(T)$ is an ideal (by Lemma 2(i)). This in turn shows by the closed graph theorem that T is bounded; actually $Tf = (T\delta_0) * f$ and $\|T\|_{1,1} = \|T\delta_0\|_1$. Now let $x_0 \in \hat{G}$. By density of $D(T)$ in L_1 there is $f \in D(T)$ satisfying $\hat{f}(x_0) \neq 0$. Without restriction we may assume that for some open neighborhood U of x_0

$$\Re\left(\hat{f}(x)\right) \geq \frac{1}{2}\hat{f}(x_0) > 0 \qquad (x \in U).$$

According to Theorems 2.6.1 and 2.6.2 in [12] we may choose a relatively compact neighborhood V_0 of x_0 and a function $g \in L_1$ such that $V_0 \subseteq U$, and

$$\hat{g}|_{U^c} = 0, 0 \leq \hat{g} \leq 1, \qquad \text{and} \qquad \hat{g}(x) = 1 \text{ for } x \in V_0.$$

Hence letting $f_0 := f * g \in D(T)$, we obtain $\Re(f_0) = \Re(\hat{f}\hat{g}) \geq 0$. Moreover,

$$\Re(\widehat{f_0}(x)) = \Re(\hat{f}(x)) \geq \frac{1}{2}\hat{f}(x_0) \qquad \text{on } V_0.$$

By compactness of $\hat{G}$ (cf., e.g., [12, Thm. 1.2.5]) there is a finite set of points $x_1, \ldots, x_r \in \hat{G}$, an open, relatively compact cover $V_1, \ldots, V_n$ of $\hat{G}$ with $x_i \in V_i$, and functions $f_1, \ldots, f_n \in D(T)$ such that

$$\Re(f_i) \geq 0 \qquad \text{and} \qquad \Re(\hat{f}_i(x)) \geq \frac{1}{2}\hat{f}_i(x_i) > 0 \qquad \text{on } V_i.$$

For $f := \sum_{i=1}^{n} f_i$ this yields $\inf\{|\hat{f}(y)| : y \in G\} > 0$, and therefore Lemma 3.7.2 in [14] implies that f has an inverse w.r.t. convolution, so $\delta_0 = g * f \in L_1 * D(T) \subseteq D(T)$ for some $g \in L_1$.

The final assertion of (ii) is the well-known characterization of bounded translation invariant operators on L_1; see, e.g., [7, Theorem and Corollary 0.1.1].

(iii) It is well known that a continuous function need not have an absolutely convergent Fourier series. So this part is a consequence of (ii). ∎

In Section 5 we will give a number of sufficient conditions for the group $(\mathbb{R}^n, +)$. In general, however, it is a rather hard problem for a given function to decide whether it is the symbol of a translation invariant operator on L_1. Of course, this reflects the similar situation in the bounded case.

In contrast to this and like the bounded case, we have a satisfying characterization of translation invariant operators on L_2 in the next theorem. Before we state it we introduce the following

Notation. If $1 \le p \le \infty$ and ϕ is measurable we put

$$D_p(\phi) := \{f \in L_p : \phi f \in L_p\}.$$

THEOREM 2. (i) *If $T \in \mathcal{T}(L_2)$ then there is a measurable $m_T: \hat{G} \to \mathbb{C}$ with $\widehat{Tf} = m_T \hat{f}$, for all $f \in D(T)$, and m_T is uniquely determined locally a.e..*

(*ii*) *If $\phi : \hat{G} \to \mathbb{C}$ is measurable then*

$$T : \mathcal{F}^{-1} D_2(\phi) \subseteq L_2 \to L_2 \qquad \textit{defined by } Tf = \mathcal{F}^{-1}(\phi \hat{f}),$$

is a translation invariant operator.

Proof. (ii) It is well known that the multiplication operator

$$M_\phi : D_2(\phi) \subseteq L_2(\hat{G}) \to L_2(\hat{G}), \qquad \text{given by } M_\phi f := \phi f,$$

is densely defined and closed. Since the Fourier transform is an isometry on L_2 the assertion is evident.

(i) The idea of this proof is to reduce it to the bounded case for the resolvent. Of course, T need not have a nonempty resolvent set. However, by a theorem of von Neumann (cf. [6, Thm. V.-3.24]) $S := T^*\overline{T}$ is densely defined, positive, selfadjoint, and $D(T^*\overline{T})$ is a core for $\overline{T}$.

Since S is positive its spectrum is contained in $\mathbb{R}_{\ge 0}$, so $(1 + S)^{-1}$ is a bounded translation invariant operator on L_2. Hence there is a unique $\phi \in L_\infty(\hat{G})$, satisfying $\mathcal{F}[(1 + S)^{-1} f] = \phi \hat{f}$ for all $f \in L_2$. Moreover, $\|\phi\|_\infty = \|(1 + S)^{-1}\|_{2,2} \le 1$.

Now, Corollary 4.1.1 in [7] in our situation shows that there is a measurable set $\Omega \subseteq \hat{G}$ such that

$$\overline{\mathcal{F}(D(S))} = \overline{\mathcal{F}(\mathrm{Ran}(1 + S)^{-1})} = 1_\Omega \cdot L_2(\hat{G}).$$

By density of $D(S)$, $\hat{G}\backslash\Omega$ is $\lambda_{\hat{G}}$-locally null and hence $\phi(x) \neq 0$ locally almost everywhere.

Define $m_S(x) := 1/\phi(x) - 1$, whenever $\phi(x) \neq 0$ and $m_S(x) = 0$ elsewhere. Then m_S is measurable, $m_S \geq 0$ locally a.e., and for any $f \in D(S) = \mathrm{Ran}(1 + S)^{-1}$ there exists $g \in L_2$ such that $\hat{f} = \phi\hat{g}$, showing that

$$\widehat{Sf} = \hat{g} - \hat{f} = \frac{1}{\phi}\phi\hat{g} - \hat{f} = m_S\hat{f},$$

almost everywhere.

Now, $D(\overline{T}) \supseteq D(T^*\overline{T}) = \mathrm{Ran}((1 + T^*\overline{T})^{-1})$, so the operator $\overline{T}(1 + T^*\overline{T})^{-1}$ is well defined on $D(\overline{T}(1 + T^*\overline{T})^{-1}) = L_2$ and is bounded, by the closed graph theorem. Since it obviously commutes with translations it has an $L_\infty(\hat{G})$-symbol ψ.

Let $m_T := \psi \cdot (1 + m_S)$, which clearly is measurable. Thus

$$\widehat{\overline{T}f} = \mathcal{F}(\overline{T}(1 + T^*\overline{T})^{-1}(1 + T^*\overline{T})f) = m_T\hat{f}$$

for all $f \in D(T^*\overline{T})$. Since, by von Neumann's theorem $D(T^*\overline{T})$ is a core, this formula extends to all of $D(T)$.

To prove uniqueness, let $\Omega := \{x \in \hat{G} : m_1(x) \neq m_2(x)\}$, where m_1 and m_2 are symbols of the same translation invariant operator T. So $1_\Omega \cdot (m_1 - m_2)\hat{f} = 0$ a.e. for all $f \in D(T)$. Therefore $1_\Omega \cdot \hat{f} = 0$ and

$$L_2(\hat{G}) = \overline{\mathcal{F}(D(T))} \subseteq 1_{\hat{G}\backslash\Omega} \cdot L_2(\hat{G}),$$

showing that Ω is $\lambda_{\hat{G}}$-locally null. ∎

Notes. The equivalence "(i) $\Leftrightarrow$ (ii)" of Theorem 1 has been proved by Babalola [1, Thm. 1.5] for general locally compact groups (not necessarily Abelian). On the other hand, equivalence "(ii) $\Leftrightarrow$ (iii)" is due to Wood [15, Thm. 2.1] for the case of general commutative Banach algebras. Obviously, if G is Abelian then these results imply Theorem 1.

Both authors define their object by the property in (ii) of Theorem 1 and dub it "unbounded multiplier." Usually, the term "multiplier" refers to a Banach algebra, the convolution algebra L_1 in the present case. However, in general, any other L_p space is not a convolution algebra, so we prefer the term "translation invariant operator" instead.

Though we state our results for the Abelian case, some of them remain true for non-Abelian groups.

In particular, parts (i) and (ii) of Lemma 2 extend to this more general setting. We have designed the proofs to remain valid in this case too. In particular Lemma 2(i), which gives "(i) $\Rightarrow$ (ii)" of Theorem 1, provides another (shorter) proof of Babalola's theorem (the proof of "(ii) $\Rightarrow$ (i)" of Theorem 1 is easy).

With a slight modification, Lemma 1 holds for non-Abelian groups too. (Assume that B is a subset of $L_q \cap \kappa L_q$ and, in case $r = \infty$, also that $A \subseteq L_p \cap \kappa L_p$.) The proof is essentially the same as in the Abelian case.

4. The Class of Translation Invariant Operators

In this section we will discuss a kind of "interpolation and duality" for translation invariant operators, uniqueness of closed extension, and algebras of translation invariant operators.

"Interpolation and duality" could be paraphrased by the chain

$$\mathcal{T}(L_1) \subseteq \mathcal{T}(L_p) \subseteq \mathcal{T}(L_q) \subseteq \mathcal{T}(L_2),$$

where $1 \leq p \leq 2$ and $p \leq q \leq p'$. Actually, we are only able to prove this if $\mathcal{T}(\cdot)$ is replaced by a slightly smaller class.

Recall that the corresponding assertion for bounded translation invariant operators (i.e., $\mathcal{T}$ replaced by $\mathcal{T}_b$) is well known (see, e.g., [7, Thm. 1.1.4]).

THEOREM 3. *Let $1 \leq p \leq 2$, $p \leq q \leq p'$, $q < \infty$, and let $T \in \mathcal{T}(L_p)$ be such that $D(\overline{T}) \cap L_1$ is dense in L_p. Then $\overline{T}|_D \in \mathcal{T}(L_q)$ for $D :=$ $\operatorname{lin}[C_c(G) * (D(\overline{T}) \cap L_1)]$.*

Proof. We first show that if $1 \leq p_1 < r < p_2 < \infty$ and if $S \in \mathcal{T}(L_{p_1}) \cap \mathcal{T}(L_{p_2})$ then $S \in \mathcal{T}(L_r)$. Note that everything is clear except closability. To prove that S is closable on L_r it suffices to show that $D((S_r)^*)$ is dense in $L_{r'}$ (subscript of S refers to the space it is considered to operate in). By reflexivity of L_{p_2}, $D((S_{p_2})^*)$ is dense in $L_{p_2'}$, so $\operatorname{lin}[C_c(G) * D((S_{p_2})^*)]$ is a core for $(S_{p_2})^*$ by Lemma 2(ii). Let $f \in C_c * D((S_{p_2})^*)$, $g \in D(S)$. Then

$$(S_{p_2})^* f \in C_c * L_{p_2'} \subseteq L_{r'},$$

by Lemma 2(i), and

$$\langle (S_{p_2})^* f, g \rangle = \langle f, S_{p_2} g \rangle = \langle f, S_r g \rangle,$$

which yields that $\operatorname{lin}[C_c * D((S_{p_2})^*)] \subseteq D((S_r)^*)$. Lemma 1 shows that $\operatorname{lin}[C_c * D((S_{p_2})^*)]$, hence also $D((S_r)^*)$, is dense in $L_{r'}$; therefore S_r is closable.

Now, in case $p = 1$, this together with Theorems 1 and 2 yields that $\overline{T}|_D \in \mathcal{T}(L_r)$ for $1 \leq r \leq 2$. Hence we may assume that $1 < p \leq 2$, so

$1 < p \leq p' < \infty$ and thus $D = \text{lin}[C_c * (D(\overline{T}) \cap L_1)]$ is dense in $L_{p'}$. By Lemma 2(iii) one has $\overline{T}|_D \subseteq \kappa T^* \kappa$. So $\overline{T}|_D$ is closable on $L_{p'}$.

This shows that $\overline{T}|_D \in \mathcal{T}(L_p) \cap \mathcal{T}(L_{p'})$; hence by what we have proved $\overline{T}|_D \in \mathcal{T}(L_q)$. ∎

An inspection of the proof shows that we may replace $C_c(G)$ in the definition of D by any other dense subspace of $L_1 \cap L_r$, as long as $1 + 1/p' \geq 1/p + 1/r$.

Note that if $q = p$ in the preceding theorem, in general, we only know that $\overline{T}$ extends the closure of $\overline{T}|_D$. Actually, $\overline{T} = \overline{\overline{T}|_D}$ if and only if $D(\overline{T}) \cap L_1$ is a core for T, as can be seen from Lemma 2(ii).

The following corollary is particularly useful in applications.

COROLLARY 2. *Let $T \in \mathcal{T}(L_1)$ and $1 \leq p < \infty$. Assume that $D(T) \subseteq L_1 \cap L_p$ is also dense in L_p and mapped to $L_1 \cap L_p$ by T. Then $T \in \mathcal{T}(L_p)$.*

Proof. By the preceding theorem we know that $\overline{T}|_D \in \mathcal{T}(L_p)$, where $D := \text{lin}[C_c * D(\overline{T})]$. To prove the assertion it suffices to show that the closure of $\overline{T}|_D$ on L_p extends T. But for any $f \in D(T)$ we may choose a sequence $(g_n) \subseteq C_c$ such that

$$g_n * f \to f \qquad \text{and} \qquad \overline{T}|_D(g_n * f) = g_n * Tf \to Tf$$

in L_p, proving that $T \subseteq \overline{\overline{T}|_D}$. ∎

Observe that any $T \in \mathcal{T}(L_1)$ admits a restriction which satisfies the assumptions of the preceding corollary ($\text{lin}[C_c * D(\overline{T})]$ is such a domain).

Our next topic will be uniqueness of the closed extension. We are going to prove that translation invariant operators on L_1 and L_2 have a uniquely determined closed extension.

In general closable operators may have different closed extensions. An example is given by the operator $C_c^2(]0, 1[) \ni f \mapsto f''$ on $L_2(0, 1)$ which has different closed extensions corresponding to different boundary conditions (cf. [11, Example VIII.6.3]).

As we will see in the following Proposition 1(iii), a similar phenomenon occurs for multiplication operators. In view of this fact it is a bit surprising that the same does not happen for translation invariant operators on L_2 (Thm. 4 below).

PROPOSITION 1. (i) *There is a dense subspace D of $L_2(\mathbb{R})$ such that for any $f \in L_2(\mathbb{R})$ there is a measurable $g \notin L_2(\mathbb{R})$ satisfying*

$$gh \in L_1 \qquad \textit{and} \qquad \langle f, h \rangle = \int ghdx$$

for all $h \in D$.

(ii) *There is a continuous function* $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *and a dense subspace* D_0 *of* $L_2(\mathbb{R})$ *such that the dual operator of*

$$M_\phi^0 : D_0 \subseteq L_2 \to L_2, f \mapsto \phi f,$$

is not a multiplication operator.

(iii) *The operator* M_ϕ^0 *of* (ii) *has at least two different closed extensions, since*

$$M_\phi^1 : D_2(\phi) \subseteq L_2 \to L_2, f \mapsto \phi f,$$

($D_2(\phi)$ has been defined in Section 3), is a closed extension of M_ϕ^0 *but different from its closure.*

Proof. Define

$$D := \{f \in L_1 \cap L_2 : \int f dx = 0\} = \{f \in L_1 \cap L_2 : \hat{f}(0) = 0\}$$

Then D is dense in L_2, since the Fourier transform is an isometry onto $L_2(\mathbb{R})$. If $f \in L_2$ then $g := f + 1 \notin L_2$. Now, $h \in D$ implies $gh \in L_1$ and

$$\int ghdx - \langle f, h \rangle = \int hdx = 0,$$

proving (i).

For the proof of (ii) let $\phi(x) := 1 + |x|$ on $\mathbb{R}$ and define $D_0 := \{f \in L_1 \cap L_2 : \phi f \in L_2, \int f dx = 0\}$. Recall that the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in L_2 and every $f \in \mathcal{S}(\mathbb{R})$ satisfies $\phi f \in L_2$. Let $\psi \in C^\infty(\mathbb{R})$ with

$$0 \leq \psi \leq 1, \qquad \psi(0) = 0, \qquad \psi(x) = 1 \quad \text{if } |x| \geq 1.$$

Define $\psi_\delta(x) := \psi(x/\delta)$ for $\delta > 0$. Clearly, for any $f \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}^{-1}(\psi_\delta \hat{f}) \in \mathcal{S}(\mathbb{R}) \subseteq L_1 \cap L_2,$$

and

$$\int \mathcal{F}^{-1}(\psi_\delta \hat{f}) dx = \psi_\delta(0)\hat{f}(0) = 0,$$

which shows that $\mathcal{F}^{-1}(\psi_\delta \hat{f}) \in D_0$. If $g \in L_2$, $\varepsilon > 0$ there is $f \in \mathcal{S}(\mathbb{R})$ with $\|f - g\|_2 \leq \varepsilon/2$. Then

$$\|f\psi_\delta - g\|_2 \leq \|f - g\|_2 + \|f - \psi_\delta f\|_2 \leq \frac{\varepsilon}{2} + \sqrt{2\delta}\, \|f\|_\infty \leq \varepsilon,$$

if δ is suitably chosen. This shows that D_0 is dense.

Now, the dual of M_ϕ^0 must be an extension of M_ϕ^1 defined in (iii) above. So if $(M_\phi^0)^*$ is a multiplication operator, then it must be multiplication by ϕ. On the other hand, let $g := 1/\phi$ and observe that $g \in L_2$. It follows that for $f \in D_0$

$$\langle g, M_\phi^0 f\rangle = \int g\phi f dx = \int f dx = 0$$

and thus $(M_\phi^0)^* g = 0 \neq 1 = \phi g$, proving (ii).

To prove (iii) note that the duals of two operators are the same if their closures coincide. So by (ii) it suffices to prove that M_ϕ^1 is selfdual, i.e., $(M_\phi^1)^* = M_\phi^1$. Obviously $(M_\phi^1)^* \supseteq M_\phi^1$. In order to prove the reverse inclusion, observe that

$$\langle (M_\phi^1)^* g, f\rangle = \langle g, M_\phi^1 f\rangle = \int g\phi f dx$$

for any $g \in D((M_\phi^1)^*)$, $f \in D_2(\phi)$. Hence

$$\int ((M_\phi^1)^* g - \phi g) f dx = 0.$$

As $D_2(\phi)$ contains all characteristic functions of sets of finite measure, it easily follows that

$$\phi g = (M_\phi^1)^* g \in L_2,$$

proving that $g \in D_2(\phi)$ and thus $(M_\phi^1)^* \subseteq M_\phi^1$. ∎

THEOREM 4. *Let $1 \leq p < \infty$ and $T_1, T_2 \in \mathcal{T}_c(L_p)$ with $T_1 \subseteq T_2$. If $D(T_i) \cap L_1$ is a core for T_i $(i = 1, 2)$ or $p = 2$ then $T_1 = T_2$.*

Proof. In case $p \neq 2$,

$$D := \mathrm{lin}[(D(T_1) \cap L_1) * (D(T_2) \cap L_1)] \subseteq D(T_1) \cap D(T_2)$$

is a core for T_2, by Lemma 2(ii). Thus T_2 is the closure of $T_2|_D$. Since $T_2|_D \subseteq T_1 \subseteq T_2$, we are done in this case.

Now assume that $p = 2$. Let ϕ be the symbol of T_2. Then $D_1 := \mathcal{F}(D(T_1)) \subseteq \mathcal{F}(D(T_2)) \subseteq D_2(\phi)$; hence all we need to show is that D_1 is a core for M, the multiplication operator on $L_2(\hat{G})$ associated with ϕ and domain $D(M) = D_2(\phi)$. This in turn is proved, once we know that the duals of $M|_{D_1}$ and M are equal. Indeed, if $(M|_{D_1})^* = M^*$ then

$$M = M^{**} = (M|_{D_1})^{**} = \overline{M|_{D_1}}.$$

Since $M|_{D_1} \subseteq M$ we have $(M|_{D_1})^* \supseteq M^*$, so we have to prove the reverse inclusion. Letting $g \in D((M|_{D_1})^*)$ we know that

$$\int (g\phi - (M|_{D_1})^*g) f d\lambda_{\hat{G}} = \langle g, M|_{D_1}f\rangle - \langle (M|_{D_1})^*g, f\rangle = 0$$

for all $f \in D_1$. Once we have shown that $g\phi - (M|_{D_1})^*g = 0$ $\lambda_{\hat{G}}$-locally a.e. we know that $g \in D_2(\phi) = D(M^*)$ and the proof is complete.

So assume that for some measurable h we have

$$hf \in L_1 \qquad \text{and} \qquad \int hf d\lambda_{\hat{G}} = 0$$

for all $f \in D_1$. Since $1_A * 1_B \in \mathcal{F}(L_1)$ (cf. [12, Thm. 1.6.3]) for any A, $B \subseteq \hat{G}$ of finite measure, we have $(1_A * 1_B)D_1 \subseteq D_1$ by Lemma 2(i). If $\lambda_{\hat{G}}(A) < \infty$, choosing suitable neighborhoods A_n we find that, letting $\gamma_n := 1/\lambda_{\hat{G}}(A_n)$,

$$1_A * \gamma_n 1_{A_n} \to 1_A$$

in L_2-norm (cf. [4, Thm. 20.15]). Taking a suitable subsequence, we may assume $\lambda_{\hat{G}}$-a.e. convergence and the estimate

$$\|1_A * \gamma_n 1_{A_n}\|_\infty \leq \|1_A\|_\infty \gamma_n \|1_{A_n}\|_1 = 1$$

yields

$$\|(1_A * \gamma_n 1_{A_n})hf - hf1_A\|_1 \to 0.$$

This shows $\int h1_A f d\lambda_{\hat{G}} = 0$ for all $f \in D_1$ and $\lambda_{\hat{G}}(A) < \infty$. Approximating an arbitrary measurable A by $A \cap A_n$, where $A_n := \{x \in \hat{G} : |f(x)| \geq 1/n\}$, we see that $\int h1_A f d\lambda_{\hat{G}} = 0$ for all measurable $A \subseteq \hat{G}$ and $f \in D_1$. Since simple functions are dense in L_∞ we conclude

$$\int hgf d\lambda_{\hat{G}} = 0$$

for all $g \in L_\infty$ and $f \in D_1$.

Finally, if $\lambda_{\hat{G}}(A) < \infty$ we let $g := (1 + |h|)^{-1}1_A$, so $g \in L_\infty \cap L_2$, and, in particular, $hg \in L_2$. Density of D_1 yields $1_A h = 0$, whenever $\lambda_{\hat{G}}(A) < \infty$. Hence $h = 0$ and the proof is complete. ∎

We remark that uniqueness of the closed extension on L_1 has been obtained in [15].

The final topic of this section is to show that, in a canonical way, the set of closed translation invariant operators on L_1 and L_2 becomes an algebra.

In general, the domain of the sum of two closed operators may be $\{0\}$. It may also happen that the sum of two closed operators with the same domain is no longer closable. And even if the sum is densely defined and closable the closed extension need not be unique, as we noted above. This nonuniqueness would cause the algebraic laws to be violated.

In contrast to this the next theorem shows that translation invariant operators, more precisely $\mathcal{T}_c(L_1)$ and $\mathcal{T}_c(L_2)$, do not show this rather pathological behavior.

THEOREM 5. *Let $p \in \{1, 2\}$. On $\mathcal{T}_c(L_p)$ define the operations*

$$T +_{\mathcal{T}} S := \overline{T + S}, \qquad T \circ_{\mathcal{T}} S := \overline{T \circ S}, \qquad \alpha \bullet_{\mathcal{T}} T := \overline{\alpha T}.$$

Then $(\mathcal{T}(L_p), +_{\mathcal{T}}, \circ_{\mathcal{T}}, \bullet_{\mathcal{T}})$ is a commutative algebra over $\mathbb{C}$.

Proof. For the case $p = 1$ we note that $D(T) * D(S) \subseteq D(T + S) \cap D(T \circ S)$ so the operations $+$ and $\circ$ are defined on a common core of S and T. On this domain (or a similar one, if more than two operators are involved), the validity of the algebraic laws is easily checked. Since the compound operators, $T + S$ and $T \circ S$, are given by symbols, they are closable (by Theorem 1) with uniquely determined closed extensions.

In case $p = 2$ note that

$$|m_T|, |m_S|, |m_T + m_S|, |m_T m_S| \leq (1 + |m_T|)(1 + |m_S|).$$

Hence the subspace $\mathcal{F}^{-1}(D_2((1 + |m_T|)(1 + |m_S|)))$ is contained in $D(T + S) \cap D(T \circ S)$. The remainder of the proof is essentially the same as in case $p = 1$. ∎

5. EXAMPLES

Now we will make good for the lack of examples so far. In order to do so we restrict ourselves to the three "classical" groups $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{T}$. In view of Theorem 2 the case $p = 2$ is trivial, so we will concentrate on examples where $p \neq 2$.

EXAMPLE 1. *Consider the operator $T: D(T) \subseteq L_p(\mathbb{Z}) \to L_p(\mathbb{Z})$ defined by $\widehat{Tf} = m_T \hat{f}$, where $m_T(z) = (1 - z)^{-1}$ and $D(T) = \{f \in L_p : m_T \hat{f} \in \mathcal{F}(L_p)\}$. Then T is not a translation invariant operator on L_1 but T is an unbounded translation invariant operator on any L_p if $1 < p < \infty$.*

Proof. If $T \in \mathcal{T}(L_p)$, $1 \le p < \infty$, then it must be unbounded because otherwise $T \in \mathcal{T}_b(L_p) \subseteq \mathcal{T}_b(L_2)$ would imply boundedness of the symbol, which is not the case. Moreover, since $\mathcal{T}(L_1) = \mathcal{T}_b(L_1)$ by Corollary 1(ii), this shows that $T \notin \mathcal{T}(L_1)$.

So it remains to show that T is densely defined and closable. Let $S \in \mathcal{T}_b(L_p)$ be defined by $Sf = f - \tau_{-1}f$. Obviously $TS = \mathrm{id}$. Since S is bounded T is closed, and it is densely defined if Ran S is dense. To prove the latter, let $g \in L_{p'}$ be such that $\langle g, Sf\rangle = 0$ for all $f \in L_p$. Then

$$0 = \langle g, S\delta_k\rangle = g(k) - g(k-1)$$

for all $k \in \mathbb{Z}$. Hence $g \equiv g(0)$, which shows that $g = 0$, since $p' < \infty$. ∎

EXAMPLE 2. *Consider the operator T of conjugation, i.e., up to some constant, $Tf = \mathcal{F}^{-1}[n \mapsto \operatorname{sgn}(n)\hat{f}(n)]$.*

T is bounded on $L_p(\mathbb{T})$, whenever $1 < p < \infty$ (theorem of M. Riesz; see, e.g., [16, VII Sect. 2]). *It is also well known that T is not bounded on L_1 (loc. cit.). Corollary* 1, *however, shows that $T \in \mathcal{T}(L_1) = \mathcal{T}(L_2)$.*

Finally, we will discuss translation invariant operators with respect to the group $(\mathbb{R}, +)$. We will give some sufficient criteria for a function ϕ on $\mathbb{R}$ to be the symbol of a translation invariant operator on L_1.

PROPOSITION 2. *Each of the following conditions implies $\phi \in \mathcal{F}\mathcal{T}(L_1)$.*

(i) *For any $\xi \in \mathbb{R}$ one has $\phi\hat{f} \in \mathcal{F}(L_1)$ for some $f \in L_1$ with $\hat{f}(\xi) \neq 0$.*

(ii) $\phi \in C^{1/2+\varepsilon}(\mathbb{R})$, *i.e., ϕ is $(\frac{1}{2} + \varepsilon)$-Hölder continuous for some $\varepsilon > 0$.*

(iii) $\phi(x) = |x|^\alpha$, $\alpha > 0$.

Proof. (i) Define $D := \{f \in L_1 : \phi\hat{f} \in \mathcal{F}L_1\}$ which is obviously a translation invariant subspace. Wiener's theorem (see, e.g., [12, Thm. 7.2.4]) shows, in view of the assumption in (i), that the closed ideal $\overline{D}$ must be L_1.

(ii) Clearly, $C_c^\infty \subseteq \mathcal{F}(L_1)$, so by (i) it suffices to prove that $\phi f \in \mathcal{F}(L_1)$ for any $f \in C_c^\infty$. But in this case $\phi f \in C_c^{1/2+\varepsilon}$, and $C_c^{1/2+\varepsilon}$ embeds continuously into the Sobolev space H^α, for some $\alpha > \frac{1}{2}$. But since $H^\alpha \subseteq \mathcal{F}(L_1)$ by Theorem 7.9.3 in [5] we are done.

(iii) This follows from Example 9.23(5) in [3] and Theorem 5. ∎

Next we give an example to show that boundedness of a symbol in $\mathcal{F}\mathcal{T}(L_1)$, in general, is not sufficient to imply that the associated translation invariant operator is bounded.

EXAMPLE 3. *The function* $\phi(x) := \sin(x^2)$ *on* $\mathbb{R}$ *is the symbol of an unbounded translation invariant operator on* $L_1(\mathbb{R})$.

Proof. Proposition 2(ii) shows that ϕ is the symbol for some $T \in \mathcal{T}(L_1)$. If $T \in \mathcal{T}_b(L_1)$ then $\phi = \hat{\mu}$, for some $\mu \in M(\mathbb{R})$, and ϕ would have to be uniformly continuous, which is not the case. ∎

Finally, let us make some remarks about the n-dimensional case. Of course, Proposition 2(i) holds on any locally compact Abelian group. On $\mathbb{R}^n$ (ii) remains true if one replaces $C_c^{1/2+\varepsilon}$ by $C_c^{n/2+\varepsilon}$ (cf. [5, Thm. 7.9.3]). To find examples beyond this class, observe that if $\phi_m \in \mathcal{F}\mathcal{T}(L_1(\mathbb{R}^m))$ and $\phi_{n-m} \in \mathcal{F}\mathcal{T}(L_1(\mathbb{R}^{n-m}))$ then $\phi_m \otimes \phi_{n-m} \in \mathcal{F}\mathcal{T}(L_1(\mathbb{R}^n))$.

We note that there is an interesting class ("derivations") of translation invariant operators which generalize first-order partial differential operators to general locally compact Abelian groups. It will be studied in a forthcoming paper.

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